

# Two Interesting Properties of the Exponential Distribution

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## Abstract

Let  $X_1, X_2, \dots, X_n$  be  $n$  independent and identically distributed random variables, here  $n \geq 2$ . Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the order statistics of  $X_1, X_2, \dots, X_n$ . In this note we proved that: (I) If  $X_1, X_2, \dots, X_n$  are exponential random variables with parameter  $c > 0$ , then the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  is strictly increasing in  $k$  from 1 to  $m$ , and then is strictly decreasing in  $k$  from  $m$  to  $n-t$ , here  $t$  is a fixed integer between 1 and  $n-3$ , and  $m = (n-t)/2$  if  $n-t$  is even,  $m = (n-t+1)/2$  if  $n-t$  is odd. We also proved that if  $t = n-2$ , then the "correlation coefficient" between  $X_{(1)}$  and  $X_{(n-1)}$  is greater than the "correlation coefficient" between  $X_{(2)}$  and  $X_{(n)}$ . (II) The "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  for the exponential random variables is always less than the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  for the uniform random variables for all  $k$  and  $t$  such that  $k+t \leq n$ . A combinatorial identity is also given as a bi-product.

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Let  $X_1, X_2, \dots, X_n$  be  $n$  independent and identically distributed exponential random variables with parameter  $c > 0$ , here  $n \geq 2$ . Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the order statistics of  $X_1, X_2, \dots, X_n$ . Without loss of generality, we can and assume that  $c = 1$ . The joint probability density function of  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  is  $f(x_{(1)}, x_{(2)}, \dots, x_{(n)}) = n! \exp\{-[x_{(1)} + x_{(2)} + \dots + x_{(n)}]\}$ , here  $0 \leq x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)} < \infty$ . Now let  $X_{(1)} = Y_1$ ,  $X_{(2)} = Y_1 + Y_2$ , ...,  $X_{(n)} = Y_1 + Y_2 + \dots + Y_n$ . Then the joint probability density function of  $Y_1, Y_2, \dots, Y_n$  is  $g(y_1, y_2, \dots, y_n) = n e^{-ny_1} (n-1) e^{-(n-1)y_2} \dots 2 e^{-2y_{n-1}} e^{-y_n}$ , where  $0 \leq y_i < \infty$  for all  $i = 1, 2, \dots, n$ . It is easy to see that  $Y_1, Y_2, \dots, Y_n$  are mutually independent and  $Y_i$  is an exponential random variable with parameter  $1/(n+1-i)$  for all  $i = 1, 2, \dots, n$ . Since  $X_{(k)} = \sum_{i=1}^k Y_i$ ,  $E(X_{(k)}) = E(\sum_{i=1}^k Y_i) = \sum_{i=1}^k \frac{1}{(n+1-i)} = \sum_{i=n+1-k}^n \frac{1}{i}$  and  $Var(X_{(k)}) = Var(\sum_{i=1}^k Y_i) = \sum_{i=1}^k \frac{1}{(n+1-i)^2} = \sum_{i=n+1-k}^n \frac{1}{i^2}$  for all  $k = 1, 2, \dots, n$ . Also  $Cov(X_{(k)}, X_{(k+t)}) = Cov(\sum_{i=1}^k Y_i, \sum_{i=1}^k Y_i + \sum_{i=k+1}^{k+t} Y_i) = Cov(\sum_{i=1}^k Y_i, \sum_{i=1}^k Y_i) + Cov(\sum_{i=1}^k Y_i, \sum_{i=k+1}^{k+t} Y_i) = Var(X_{(k)}) + 0 = Var(X_{(k)})$  since  $\sum_{i=1}^k Y_i$  and  $\sum_{i=k+1}^{k+t} Y_i$  are independent, here  $1 \leq t \leq n - k$ .

$E(X_{(n)}^2) = \int_0^\infty x^2 n (1 - e^{-x})^{n-1} e^{-x} dx = n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \frac{2}{(j+1)^3} = 2 [\sum_{i=1}^n \binom{n}{i} \frac{(-1)^{i+1}}{i^2}] = Var(X_{(n)}) + [E(X_{(n)})]^2 = \sum_{i=1}^n \frac{1}{i^2} + [\sum_{i=1}^n \frac{1}{i}]^2$ . Therefore, we have the following combinatorial identity

$$(1) \quad \sum_{i=1}^n \binom{n}{i} \frac{(-1)^{i+1}}{i^2} = \frac{1}{2} \left\{ \sum_{i=1}^n \frac{1}{i^2} + \left[ \sum_{i=1}^n \frac{1}{i} \right]^2 \right\}.$$

The following combinatorial identity

$$(2) \quad \sum_{i=1}^n \binom{n}{i} (-1)^{i+1} \frac{1}{i} = \sum_{i=1}^n \frac{1}{i}.$$

is known. However, it can be derived simply by computing  $E(X_{(n)}) = \int_0^\infty nx(1 - e^{-x})^{n-1} e^{-x} dx = n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \frac{1}{(j+1)^2} = \sum_{i=1}^n \binom{n}{i} (-1)^{i+1} \frac{1}{i} = \sum_{i=1}^n \frac{1}{i}$  since  $E(X_{(n)}) = \sum_{i=1}^n \frac{1}{i}$ .

The combinatorial identity (1) might be new.

Let  $\rho_{k,t}$  be the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$ , where  $1 \leq k \leq n-t$  and  $t$  is a fixed positive integer such that  $1 \leq t$  and  $k+t \leq n$ . First we will prove that  $\rho_{k,t}$  is strictly increasing in  $k$  from 1 to  $m$  and then is strictly decreasing in  $k$  from  $m$  to  $n-t$ . It is easy to check that  $\rho_{k,t}^2 = [\sum_{i=n+1-k}^n \frac{1}{i^2}] / [\sum_{i=n+1-k-t}^n \frac{1}{i^2}]$  since  $Cov(X_{(k)}, X_{(k+t)}) = Var(X_{(k)})$ . Since  $t$  is fixed, we will let  $h(k) = \rho_{k,t}^2$  and are interested in the function  $h(k)$  for  $k$  from 1 to  $n-t$ . First we give a few examples.

Example 1:  $n = 5$ .

$$t = 1, h(1) \approx 0.390, h(2) \approx 0.480, h(3) \approx 0.461, h(4) \approx 0.317.$$

$$t = 2, h(1) \approx 0.187, h(2) \approx 0.221, h(3) \approx 0.146.$$

$$t = 3, h(1) \approx 0.086, h(2) \approx 0.070.$$

$$t = 4, h(1) \approx 0.027.$$

Example 2:  $n = 6$ .

$$t = 1, h(1) \approx 0.410, h(2) \approx 0.520, h(3) \approx 0.540, h(4) \approx 0.491, h(5) \approx 0.329.$$

$$t = 2, h(1) \approx 0.213, h(2) \approx 0.281, h(3) \approx 0.265, h(4) \approx 0.162.$$

$$t = 3, h(1) \approx 0.115, h(2) \approx 0.138, h(3) \approx 0.087.$$

$$t = 4, h(1) \approx 0.056, h(2) \approx 0.045.$$

$$t = 5, h(1) \approx 0.019.$$

Example 3:  $n = 8$ .

$$t = 1, h(1) \approx 0.434, h(2) \approx 0.565, h(3) \approx 0.615, h(4) \approx 0.624, h(5) \approx 0.599,$$

$$h(6) \approx 0.526, h(7) \approx 0.345.$$

$$t = 2, h(1) \approx 0.245, h(2) \approx 0.347, h(3) \approx 0.384, h(4) \approx 0.374, h(5) \approx 0.315,$$

$$h(6) \approx 0.182.$$

$$t = 3, h(1) \approx 0.151, h(2) \approx 0.217, h(3) \approx 0.230, h(4) \approx 0.197, h(5) \approx 0.109.$$

$$t = 4, h(1) \approx 0.094, h(2) \approx 0.130, h(3) \approx 0.121, h(4) \approx 0.068.$$

$$t = 5, h(1) \approx 0.056, h(2) \approx 0.068, h(3) \approx 0.042.$$

$$t = 6, h(1) \approx 0.030, h(2) \approx 0.024.$$

$t = 7, h(1) \approx 0.010.$

Example 4:  $n = 9.$

$t = 1, h(1) \approx 0.441, h(2) \approx 0.578, h(3) \approx 0.635, h(4) \approx 0.656, h(5) \approx 0.650,$

$h(6) \approx 0.617, h(7) \approx 0.537, h(8) \approx 0.351.$

$t = 2, h(1) \approx 0.355, h(2) \approx 0.367, h(3) \approx 0.417, h(4) \approx 0.426, h(5) \approx 0.401,$

$h(6) \approx 0.331, h(7) \approx 0.188.$

$t = 3, h(1) \approx 0.162, h(2) \approx 0.241, h(3) \approx 0.271, h(4) \approx 0.263, h(5) \approx 0.215,$

$h(6) \approx 0.116.$

$t = 4, h(1) \approx 0.106, h(2) \approx 0.157, h(3) \approx 0.167, h(4) \approx 0.141, h(5) \approx 0.075.$

$t = 5, h(1) \approx 0.069, h(2) \approx 0.097, h(3) \approx 0.090, h(4) \approx 0.050.$

$t = 6, h(1) \approx 0.043, h(2) \approx 0.052, h(3) \approx 0.031.$

$t = 7, h(1) \approx 0.022, h(2) \approx 0.018.$

$t = 8, h(1) \approx 0.008.$

From these examples, we can see that for a fixed "t",  $h(k)$  is strictly increasing and then strictly decreasing for  $k$  from 1 to  $n - t$ , except that when  $t = n - 2$ , then  $h(1) > h(2)$  (when  $t = n - 2$ ,  $k$  can be 1 or 2 only).

Theorem 1:

(I) For any fixed  $t$  between 1 and  $n - 3$ ,  $h(k)$  is strictly increasing for  $1 \leq k \leq m$  and is strictly decreasing for  $m \leq k \leq n - t$ , where  $m = (n - t)/2$  if  $n - t$  is even  $= (n + 1 - t)/2$  if  $n - t$  is odd.

(II) For  $t = n - 2$ , then  $h(1) > h(2).$

Before we prove "Theorem 1", we state a "Lemma" without a proof since it is easy to check.

Lemma: Assume that  $a$ ,  $b$ ,  $c$ , and  $d$  are positive numbers,

(a) If  $a/(a+b) > c/d$ , then  $a/(a+b) > (a+c)/(a+b+d) > c/d$ .

(b) If  $a/(a+b) < c/d$ , then  $a/(a+b) < (a+c)/(a+b+d)$ .

Now we start to prove "Theorem 1".

(I) For a fixed  $t$  between 1 and  $n-3$ ,  $m \geq 2$ . We first will show that

$$h(m) = \left[ \sum_{i=n+1-m}^n \frac{1}{i^2} \right] / \left[ \sum_{i=n+1-m-t}^n \frac{1}{i^2} \right] > \frac{(n-m-t)^2}{(n-m)^2}.$$

If this is proved, then by the "Lemma",  $h(m) > h(m+1) > (n-m-t)^2/(n-m)^2$  and  $h(m+1) > (n-m-t-1)^2/(n-m-1)^2$  since  $(n-m-t)^2/(n-m)^2$  is strictly decreasing in  $m$  for a fixed " $t$ ". By this process, we will have  $h(m) > h(m+1) > \dots > h(n-t)$ , i.e.,  $h(k)$  is strictly decreasing in  $k$  from  $m$  to  $(n-t)$ . To prove that  $h(m) > (n-m-t)^2/(n-m)^2$ , let  $N = \sum_{i=n+1-m}^n \frac{1}{i^2}$  and  $D = \sum_{i=n+1-m-t}^n \frac{1}{i^2}$ . It is easy to check that

$$\begin{aligned} N &> \left[ \frac{1}{2(n+1-m)^2} + \int_{n+1-m}^n \frac{1}{x^2} dx + \frac{1}{2n^2} \right] \\ &= \frac{n^2 + (n+1-m)^2 + 2n(n+1-m)(m-1)}{2n^2(n+1-m)^2}. \end{aligned}$$

Also it is easy to check that

$$\begin{aligned} D &< \int_{n+1-m-t}^n \frac{1}{(x-\frac{1}{2})^2} dx = \frac{2}{2n+1-2m-2t} - \frac{2}{2n+1} \\ &= \frac{4(m+t)}{(2n+1)(2n+1-2m-2t)}. \end{aligned}$$

To prove that  $h(m) = N/D > (n-m-t)^2/(n-m)^2$ , it is sufficient to show that

$$\begin{aligned} &\frac{(2n+1)(2n+1-2m-2t)[n^2 + (n+1-m)^2 + 2n(n+1-m)(m-1)]}{8(m+t)n^2(n+1-m)^2} \\ &> \frac{(n-m-t)^2}{(n-m)^2} \end{aligned}$$

since  $h(m) = N/D >$

$$\frac{(2n+1)(2n+1-2m-2t)[n^2+(n+1-m)^2+2n(n+1-m)(m-1)]}{8(m+t)n^2(n+1-m)^2}.$$

Since all numbers involved are positive numbers, it is sufficient to show that

$$(3) \quad (2n+1)(2n+1-2m-2t)[n^2+(n+1-m)^2+2n(n+1-m)(m-1)](n-m)^2 \\ - 8(m+t)n^2(n+1-m)^2(n-m-t)^2 > 0.$$

There are two cases to be considered:

$$(a) \quad n = 2m + t.$$

Then to prove the inequality (3) is equivalent to prove the following inequality

$$(4) \quad (4m+2t+1)(2m+1)(m+t)^2[(2m+t)^2+(m+t+1)^2+2(2m+t)(m+t+1)(m-1)] \\ - 8(m+t)m^2(m+t+1)^2(2m+t)^2 > 0.$$

After simplification, we have the following inequality:

$$(5) \quad (16t-14)m^5 + (48t^2-8t+1)m^4 + (53t^3+24t^2+4t+4)m^3 \\ + (24t^4+24t^3+5t^2+10t+1)m^2 + (4t^5+6t^4+2t^3+8t^2+2t)m \\ + t^2(2t+1) > 0$$

since  $t \geq 1$  and  $m \geq 2$ . Hence  $h(m) > (n-m-t)^2/(n-m)^2$  when  $n = 2m + t$ .

$$(b) \quad n = 2m + t - 1.$$

Then to prove the inequality (3) is equivalent to prove the following inequality

$$(6) \quad (4m+2t-1)(2m-1)(m+t-1)^2[(2m+t-1)^2+(m+t)^2+2(2m+t-1)(m+t)(m-1)] \\ - 8(m+t)^3(2m+t-1)^2(m-1)^2 > 0.$$

After simplification, the left hand side of the inequality (6) is

$$(7) \quad (48t-14)m^5 + (144t^2-120t+47)m^4 + (156t^3-256t^2+128t-60)m^3$$

$$+ (72t^4 - 216t^3 + 155t^2 - 80t + 36)m^2 + (12t^5 - 74t^4 + 86t^3 - 42t^2 + 28t - 10)m \\ + (-8t^5 + 16t^4 - 10T63 + 5t^2 - 4t + 1).$$

We have to re-arrange (7) by using the fact that  $m \geq 2$  to show that

$$(8) \quad (48t - 14)m^5 + (144t^2 - 120t + 47)m^4 + (156t^3 - 256t^2 + 128t - 60)m^3 \\ + (72t^4 - 216t^3 + 155t^2 - 80t + 36)m^2 + (12t^5 - 74t^4 + 86t^3 - 42t^2 + 28t - 10)m \\ + (-8t^5 + 16t^4 - 10T63 + 5t^2 - 4t + 1) \geq 14(t - 1)m^5 + 26(t - 1)^2m^4 \\ + [67t(t - 1)^2 + 18(t - 1)]m^3 + 19t^2(t - 1)^2m^2 \\ + (8t^5 + 32t^4 + 81t^3 + 230t^2 + 98t + 62)m + (16t^4 + 5t^2 + 1) > 0$$

since  $t \geq 1$  and  $m \geq 2$ . Hence the inequality (6) holds and  $h(m) > \frac{(n-m-t)^2}{(n-m)^2}$ .

By the "Lemma", we can conclude both cases that  $h(m) > h(m+1) > \frac{(n-m-t)^2}{(n-m)^2} > \frac{(n-m-t-1)^2}{(n-m-1)^2}$ . Therefore,  $h(m+1) > h(m+2)$ . By this process, we have proved that  $h(k)$  is strictly decreasing in  $k$  from  $m$  to  $n - t$ , here  $t$  is a fixed integer between 1 and  $(n - 3)$ .

Now we have to prove that

$$h(m-1) = \left[ \sum_{i=n+2-m}^n \frac{1}{i^2} \right] / \left[ \sum_{i=n+2-m-t}^n \frac{1}{i^2} \right] < \frac{(n+1-m-t)^2}{(n+1-m)^2}.$$

As above let  $N = \sum_{i=n+2-m}^n \frac{1}{i^2}$  and  $D = \sum_{i=n+2-m-t}^n \frac{1}{i^2}$ . It is easy to see that

$$N < \int_{n+2-m}^{n+1} \frac{1}{(x-0.5)^2} dx = \frac{2}{(2n+3-2m)} - \frac{2}{(2n+1)} \\ = \frac{4(m-1)}{(2n+1)(2n+3-2m)}.$$

Also it is easy to check that

$$D > \frac{[n^2 + (n+2-m-t)^2 + 2n(n+2-m-t)(m+t-2)]}{[2n^2(n+2-m-t)^2]}.$$

To prove that  $h(m-1) < \frac{(n+1-m-t)^2}{(n+1-m)^2}$ , it is sufficient to show that

$$(9) \frac{8(m-1)n^2(n+2-m-t)^2}{(2n+1)(2n+3-2m)[n^2+(n+2-m-t)^2+2n(n+2-m-t)(m+t-2)]} < \frac{(n+1-m-t)^2}{(n+1-m)^2}.$$

As above, it is equivalent to show that

$$(10) (2n+1)(2n+3-2m)[n^2+(n+2-m-t)^2+2n(n+2-m-t)(m+t-2)](n+1-m-t)^2 - 8(m-1)n^2(n+2-m-t)^2(n+1-m)^2 > 0.$$

There are also two cases to be considered.

$$(c) \quad n = 2m + t.$$

Then to prove the inequality (10) is equivalent to prove the following inequality

$$(11) (4m+2t+1)(2m+2t+3)(m+1)^2[(2m+t)^2+(m+2)^2+2(2m+t)(m+2)(m+t-2)] - 8(m-1)(m+2)^2(2m+t)^2(m+t+1)^2 > 0.$$

After simplification, the inequality (11) becomes

$$(12) (48t-14)m^5 + (96t^2+242t-21)m^4 + (60t^3+444t^2+472t+20)m^3 + (12t^4+256t^3+695t^2+290t+59)m^2 + (48t^4+332t^3+282t^2+20t+44)m + (52t^4+72t^3-t^2+8t+12) > 0$$

since  $t \geq 1$  and  $m \geq 2$ .

$$(d) \quad n = 2m + t - 1.$$

Then the inequality (10) becomes

$$(13) m^2(4m+2t-1)(2m+2t+1)[(m+1)^2+(2m+t-1)^2+2(2m+t-1)(m+1)(m+t-2)]$$



$$- 8(m-1)(m+2)^2(2m+t)^2(m+t-1)^2 > 0.$$

After simplification, the inequality (13) becomes

$$(14) \quad (16t-14)m^5 + (32t^2+2t+25)m^4 + (20t^3+36t^2+52t-4)m^3 \\ + (4t^4+32t^3+53t^2-56t+2)m^2 + (8t^4+32t^3-56t^2+16t)m + 8t^2(t-1)^2 > 0$$

since  $t \geq 1$  and  $m \geq 2$ . Hence  $h(m-1) < \frac{(n+1-m-t)^2}{(n+1-m)^2}$  and  $h(m-1) < h(m)$ .

Now we have to show that  $h(k)$  is strictly increasing in  $k$  from 1 to  $m$ . Suppose not, then there exists a  $k$  such that  $h(k) \geq h(k+1)$ , where  $1 \leq k \leq m-2$  since  $h(m-1) < h(m)$ . If  $h(k) = h(k+1)$ , then  $h(k) = \frac{(n-k-t)^2}{(n-k)^2}$  and  $h(k) = h(k+1) > \frac{(n-1-k-t)^2}{(n-1-k)^2}$ . By the "Lemma", then  $h(k+1) > h(k+2) > \dots > h(m-1) > h(m)$  and we get a contradiction. If  $h(k) > h(k+1)$ , then  $h(k+1) > h(k+2) > \dots > h(m-1) > h(m)$  and we get a contradiction again. Hence  $h(k)$  is strictly increasing in  $k$  from 1 to  $m$ . The part (I) of the "Theorem 1" is proved.

To complete the proof of the "Theorem 1", now we have to prove the part (II) of the "Theorem 1".

When  $t = n-2$  and  $n \geq 3$ ,  $k$  can be 1 or 2 only. Now we will show that  $h(1) > h(2)$ .  $h(1) = \frac{1}{n^2 \sum_{i=2}^n \frac{1}{i^2}}$ , to prove  $h(1) > h(2)$ , we only need to show  $\frac{1}{n^2 \sum_{i=2}^n \frac{1}{i^2}} > \frac{1}{(n-1)^2}$ . It is easy to see that

$$(15) \quad \sum_{i=2}^n \frac{1}{i^2} < \int_2^{n+1} \frac{1}{(x-0.5)^2} = \frac{2}{3} - \frac{2}{2n+1} = \frac{4(n-1)}{3(2n+1)}.$$

If  $\frac{4(n-1)}{3(2n+1)} < \frac{(n-1)^2}{n^2}$ , then  $\sum_{i=2}^n \frac{1}{i^2} < \frac{(n-1)^2}{n^2} \cdot \frac{4(n-1)}{3(2n+1)} < \frac{(n-1)^2}{n^2}$  if  $3(2n+1)(n-1) - 4n^2 > 0$ .  $3(2n+1)(n-1) - 4n^2 = 2(n-1)^2 + (n-3) > 0$  since  $n \geq 3$ . Therefore,  $h(1) > h(2)$  and the proof of the "Theorem 1" now is complete.

By the same process, we also get an upper bound for  $h(m)$ . Hence we have the following inequality.

$$\frac{(2n+1)(2n+1-2m-2t)[n^2 + (n+1-m)^2 + 2n(n+1-m)(m-1)]}{8n^2(n+1-m)^2(m+t)} < h(m)$$

$$< \frac{8n^2(n+1-m-t)^2m}{(2n+1)(2n+1-2m)[n^2+(n+1-m-t)^2+2n(n+1-m-t)(m+t-1)]}.$$

Suppose that  $t = [nx]$ , here  $[nx]$  is the largest integer  $\leq nx$  and  $t \leq n-3$ . Since  $h(m) = \rho_{m,t}^2$  and by the "Lemma",  $h(m) < \frac{(n+1-m-t)^2}{(n+1-m)^2}$ , we have the following inequality for  $\rho_{m,t}$

$$\frac{(n-m-t)}{(n-m)} < \rho_{m,t} < \frac{(n+1-m-t)}{(n+1-m)}.$$

Substitute  $[nx]$  for  $t$ , if  $n - [nx]$  is even, we have the following inequality for  $\rho_{m,t}$

$$\frac{(n-[nx])}{(n+[nx])} < \rho_{m,t} < \frac{(n-[nx]+2)}{(n+[nx]+2)}.$$

And if  $n - [nx]$  is odd, we have the following inequality for  $\rho_{m,t}$

$$\frac{(n-[nx]-1)}{(n+[nx]-1)} < \rho_{m,t} < \frac{(n-[nx]+1)}{(n+[nx]+1)}.$$

Further more if  $[nx] = nx$  i.e.,  $nx$  is an integer, then if  $n - nx$  is even we have the following inequality for  $\rho_{m,t}$

$$\frac{(1-x)}{(1+x)} < \rho_{m,t} < \frac{(1-x+\frac{2}{n})}{(1+x+\frac{2}{n})}.$$

And if  $n - nx$  is odd we have the following inequality for  $\rho_{m,t}$

$$\frac{(1-x-\frac{1}{n})}{(1+x-\frac{1}{n})} < \rho_{m,t} < \frac{(1-x+\frac{1}{n})}{(1+x+\frac{1}{n})}.$$

If  $n$  is large, the lower bound and the upper bound are so close, and  $\rho_{m,t} \approx \frac{(1-x)}{(1+x)}$ .

In fact, if we replace  $m$  by  $k$  for  $k$  from 1 to  $n-3$ , we have upper and lower bounds for  $h(k)$  as follows:

$$(16) \quad \frac{(2n+1)(2n+1-2k-2t)[n^2+(n+1-k)^2+2n(n+1-k)(k-1)]}{8n^2(n+1-k)^2(k+t)} < h(k) \\ < \frac{8n^2(n+1-k-t)^2k}{(2n+1)(2n+1-2k)[n^2+(n+1-k-t)^2+2n(n+1-k-t)(k+t-1)]}.$$

It is very easy to compute the lower and upper bounds for  $\rho_{k,t}$  for any  $k$  and  $t$ , here  $t$  is fixed and is between 1 and  $n-3$ ,  $k+t \leq n$  even just with a hand calculator. Also the both bounds are very close to the exact value of  $\rho_{k,t}$ . However, even moderate  $n$ , it needs some software like Maple or Mathematica to compute  $\rho_{k,t}$ .

Now suppose that  $X_1, X_2, \dots, X_n$  are  $n$  independent and identically distributed uniform random variables over the interval  $[0, 1]$  here  $n \geq 2$ . Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the order statistics of  $X_1, X_2, \dots, X_n$ . It is well-known that the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  is equal to  $\sqrt{\frac{k(n+1-k-t)}{(k+t)(n+1-k)}}$ . It is easy to see that it is strictly increasing in  $k$  from 1 to  $m$  and then strictly decreasing in  $k$  from  $m$  to  $n-t$  if  $n-t$  is odd. However, if  $n-t$  is even, then it is strictly increasing in  $k$  from 1 to  $m$  and then strictly decreasing in  $k$  from  $m+1$  to  $n-t$ . For  $k=m$  and  $k=m+1$ , they are the same. It is different from the case for the exponential random variables. Further more the "correlation coefficient" between  $X_{(1)}$  and  $X_{(n-1)}$  is greater than the "correlation coefficient" between  $X_{(2)}$  and  $X_{(n)}$  for the exponential random variables, but the "correlation coefficient" between  $X_{(1)}$  and  $X_{(n-1)}$  is equal to the "correlation coefficient" between  $X_{(2)}$  and  $X_{(n)}$  for the uniform random variables, both of them are equal to  $\sqrt{\frac{2}{n(n-1)}}$ .

From our computation, we observed that the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  for the exponential random variables is always less than the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  for the uniform random variables, here  $k, t$  are positive integers and  $k+t \leq n$ ,  $n \geq 2$ . We have the following theorem.

Theorem 2:

The "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  for the exponential random variables is always less than the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  for the uniform random variables, here  $k, t$  are positive integers and  $k+t \leq n$ ,  $n \geq 2$ .

It is sufficient to show that  $h(k)$  is less than  $\frac{k(n+1-k-t)}{(k+t)(n+1-k)}$ . There are three cases to discuss.

Case I: When  $t = 1$ .

$$(17) \quad \frac{\sum_{n+1-k}^n \frac{1}{i^2}}{\sum_{n-k}^n \frac{1}{i^2}} < \frac{k(n-k)}{(k+1)(n+1-k)}.$$

It is equivalent to show that

$$(n-k)^2 \sum_{n-k}^n \frac{1}{i^2} < \frac{(k+1)(n+1-k)}{n+1}.$$

After simplification, we have  $(2n+1)(2n+1-2k) - 4(n+1)(n-k) = 2k+1 > 0$  since  $k \geq 1$ .

Case II: When  $k = 1$ .

It is equivalent to show that

$$(18) \quad \frac{1}{\sum_{n-t}^n \frac{1}{i^2}} < \frac{n(n-t)}{t+1}.$$

Since  $\sum_{n-t}^n \frac{1}{i^2} > \frac{n^2+(n-t)^2+2n(n-t)t}{2n^2(n-t)^2}$ , it is sufficient to show that  $n^2 + (n-t)^2 + 2n(n-t)t - 2n(n-t)(t+1) > 0$ . After simplification, we have  $t^2 > 0$  since  $t \geq 1$ .

From now on we will assume that  $k, t \geq 2$ . Hence  $n \geq 4$ . Recall that

$$h(k) = \frac{N}{D} = \frac{\sum_{n+1-k}^n \frac{1}{i^2}}{\sum_{n+1-k}^n \frac{1}{i^2}}.$$

$$N = \sum_{n+1-k}^n \frac{1}{i^2} < \frac{(2n+1)(2n+3-2k) + 4(n+1-k)^2(k-1)}{(2n+1)(2n+3-2k)(n+1-k)^2}.$$

And

$$D = \sum_{n+1-k-t}^n \frac{1}{i^2} > \frac{n^2 + (n+1-k-t)^2 + 2n(n+1-k-t)(k+t-1)}{2n^2(n+1-k-t)^2}.$$

Hence

$$h(k) < \frac{2n^2(n+1-k-t)^2[(2n+1)(2n+3-2k) + 4(k-1)(n+1-k)^2]}{(2n+1)(2n+3-2k)(n+1-k)^2[n^2 + (n+1-k-t)^2 + 2n(n+1-k-t)(k+t-1)]}.$$

To show  $h(k) < \frac{k(n+1-k-t)}{(k+t)(n+1-k)}$ , it is sufficient to show that

$$\frac{k(n+1-k-t)}{(k+t)(n+1-k)} > \frac{2n^2(n+1-k-t)^2[(2n+1)(2n+3-2k)+4(k-1)(n+1-k)^2]}{(2n+1)(2n+3-2k)(n+1-k)^2[n^2+(n+1-k-t)^2+2n(n+1-k-t)(k+t-1)]}.$$

From now on we will let  $n = k + t + x$ , where  $x$  is a non-negative integer. To show that

$$\frac{k(n+1-k-t)}{(k+t)(n+1-k)} > \frac{2n^2(n+1-k-t)^2[(2n+1)(2n+3-2k)+4(k-1)(n+1-k)^2]}{(2n+1)(2n+3-2k)(n+1-k)^2[n^2+(n+1-k-t)^2+2n(n+1-k-t)(k+t-1)]},$$

it is equivalent to show that

$$k(n+1-k-t)(2n+1)(2n+3-2k)(n+1-k)^2[n^2+(n+1-k-t)^2+2n(n+1-k-t)(k+t-1)] - 2(k+t)(n+1-k)n^2(n+1-k-t)^2[(2n+1)(2n+3-2k)+4(k-1)(n+1-k)^2] > 0.$$

Replace  $n$  by  $k + t + x$ , we obtain the following polynomial in  $x$

$$(19) \quad p(x) = [8k^2+8k(t-1)-8t]x^7 + [16k^3+(24+56t)k^2+8(5t^2-2t-5)k-40(t^2+t)]x^6 \\ + [8k^4+72(t+1)k^3+(144t^2+188t-6)k^2+(80t^3+36t^2-182t-74)k-(80t^3+176t^2+78t)]x^5 \\ + [8(3t+5)k^4+(120t^2+296t+114)k^3+(176t^3+480t^2+140t-100)k^2 \\ + (80t^4+144t^3-278t^2-386t-54)k-(80t^4+304t^3+298t^2+74t)]x^4$$

$$\begin{aligned}
& +[(24t^2 + 100t + 78)k^4 + (88t^3 + 436t^2 + 444t + 61)k^3 + (104t^4 + 552t^3 \\
& + 492t^2 - 172t - 144)k^2 + (40t^5 + 176t^4 - 130t^3 - 655t^2 - 338t + 5)k \\
& - (40t^5 + 256t^4 + 434t^3 + 240t^2 - 34t)]x^3 \\
& [(8t^3 + 80t^2 + 156t + 740)k^4 + (24t^4 + 272t^3 + 560t^2 + 276t - 21)k^3 + (24t^5 \\
& + 296t^4 + 576t^3 + 52t^2 - 316t - 84)k^2 + (8t^6 + 95t^5 + 58t^4 - 440t^3 - 545t^2 \\
& - 110t + 27)k - (8t^6 + 104t^5 + 294t^4 + 282t^3 + 90t^2 + 6t - 4)]x^2 \\
& +[(20t^3 + 90t^2 + 108t + 34)k^4 + (60t^4 + 276t^3 + 307t^2 + 44t - 33)k^3 \\
& + (60t^5 + 266t^4 + 198t^3 - 168t^2 - 160t - 28)k^2 + (20t^6 + 64t^5 - 89t^4 - 312t^3 \\
& - 178t^2 + 9)k - (16t^6 + 88t^5 + 140t^4 + 78t^3 + 12t^2 - 8t - 8)]x \\
& +[(12t^3 + 34t^2 + 28t + 6)k^4 + (36t^4 + 92t^3 + 53t^2 - 12t - 9)k^3 \\
& (36t^5 + 74t^4 - 2t^3 - 64t^2 - 24t)k^2 + (12t^6 + 8t^5 - 51t^4 - 66t^3 - 18t^2 - 1)k \\
& - (8t^6 + 24t^5 + 22t^4 + 6t^3 - 4t^2 - 8t - 4)] \\
& \geq [8k^2 + 8k(t - 1) - 8t]x^7 + [16k^3 + (40t^2 + 96t + 8)k - (40t^2 + 40t)]x^6 \\
& + [8k^4 + 72(t + 1)k^3 + (80t^3 + 324t^2 + 194t - 86)k - (80t^3 + 176t^2 + 78t)]x^5 \\
& + [8(3t + 5)k^4 + (120t^2 + 296t + 114)k^3 + (80t^4 + 496t^3 + 682t^2 - 106t - 254)k \\
& - (80t^4 + 304t^3 + 298t^2 + 74t)]x^4 \\
& + [(24t^2 + 100t + 78)k^4 + (88t^3 + 436t^2 + 444t + 61)k^3 \\
& + (40t^5 + 384t^4 + 974t^3 + 329t^2 - 682t - 283)k - (40t^5 + 256t^4 + 434t^3 + 240t^2 - 34t)]x^3
\end{aligned}$$

$$\begin{aligned}
& +[(8t^3 + 80t^2 + 156t + 740)k^4 + (24t^4 + 272t^3 + 560t^2 + 276t - 21)k^3 + (8t^6 + 143t^5 \\
& + 650t^4 + 712t^3t^3 - 441t^2 - 742t - 141)k - (8t^6 + 104t^5 + 294t^4 + 282t^3 + 90t^2 + 6t - 4)]x^2 \\
& +[(20t^3 + 90t^2 + 108t + 34)k^4 + (60t^4 + 276t^3 + 307t^2 + 44t - 33)k^3 + (20t^6 + 184t^5 + 443t^4 \\
& + 84t^3 - 514t^2 - 320t - 47)k - (16t^6 + 88t^5 + 140t^4 + 78t^3 + 12t^2 - 8t - 8)]x \\
& +[(12t^3 + 34t^2 + 28t + 6)k^4 + (36t^4 + 92t^3 + 53t^2 - 12t - 9)k^3 + (12t^6 + 80t^5 + 97t^4 - 70t^3 \\
& - 146t^2 - 48t - 1)k - (8t^6 + 24t^5 + 22t^4 + 6t^3 - 4t^2 - 8t - 4)] > 0
\end{aligned}$$

since  $k, t \geq 2$  and  $x$  is a non-negative integer. The proof of Theorem 2 now is complete.

Theorem 1 tells us that the "correlation coefficient" between  $X(k)$  and  $X(k+t)$  is largest when  $k = \frac{n-t}{2}$  if  $n-t$  is even, and  $k = \frac{n-t+1}{2}$  if  $n-t$  is odd, also the "correlation coefficient" between  $X(1)$  and  $X(n-1)$  is larger than the "correlation coefficient" between  $X(2)$  and  $X(n)$  for the exponential random variables. From our computation, this theorem does not hold for the random variables with the probability density function  $f(x) = 2x$  for  $0 \leq x \leq 1$ . When  $n = 3$ , the "correlation coefficient" between  $X(1)$  and  $X(2)$  is less than the "correlation coefficient" between  $X(2)$  and  $X(3)$ . Also when  $n = 7$  and  $t = 1$ , the "correlation coefficient" between  $X(k)$  and  $X(k+1)$  is largest when  $k = 4 > \frac{n-1}{2}$ . However, when  $n = 6$ , the "correlation coefficient" between  $X(k)$  and  $X(k+1)$  is largest when  $k = 3 = \frac{n-t+1}{2}$ . For the random variables with the probability density function  $f(x) = 2(1-x)$  for  $0 \leq x \leq 1$ , Theorem 1 seems to hold. So we have the following conjecture:

Conjecture I:

Theorem 1 holds if the probability density function is strictly decreasing. Theorem 1 must be modified as the "correlation coefficient" between  $X(k)$  and  $X(k+t)$  is largest when  $k = m = \frac{n-t+1}{2}$  when  $n-t$  is odd, and the "correlation coefficient" between  $X(k)$  and  $X(k+t)$  is largest when  $k = m+1 = \frac{n-t}{2} + 1$  when  $n-t$  is even if the probability density function is strictly increasing. We do not have any idea about the case that the probability density is increasing and then decreasing.

Suppose that  $Y_1, Y_2, \dots, Y_n$  be  $n$  independent and identically distributed negative exponential random variables, here  $n \geq 2$ . Let  $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$  be the order statistics of  $Y_1, Y_2, \dots, Y_n$ . Let  $Y_i = -X_{n+1-i}$  for all  $i = 1, 2, \dots, n$ , then  $X_1, X_2, \dots, X_n$  are  $n$  independent and identically distributed exponential random variables, and  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  be the order statistics of  $X_1, X_2, \dots, X_n$ . The "correlation coefficient" between  $Y_{(i)}$  and  $Y_{(i+t)}$  is the same as the "correlation coefficient" between  $X_{(n+1-i)}$  and  $X_{(n+1-i-t)}$ . So the "correlation coefficient" between  $Y_{(i)}$  and  $Y_{(i+t)}$  is strictly increasing in  $i$  from 1 to  $m$ , and is strictly decreasing in  $i$  from  $m$  to  $n-t$ , if  $n-t$  is odd and here  $m = \frac{n+1-t}{2}$ . And the "correlation coefficient" between  $Y_{(i)}$  and  $Y_{(i+t)}$  is strictly increasing in  $i$  from 1 to  $m+1$ , and is strictly decreasing in  $i$  from  $m+1$  to  $n-t$ , if  $n-t$  is even and here  $m = \frac{n-t}{2}$ . The "correlation coefficient" between  $X_{(1)}$  and  $X_{(n-1)}$  is larger than the "correlation coefficient" between  $X_{(2)}$  and  $X_{(n)}$ . So the "correlation coefficient" between  $Y_{(1)}$  and  $Y_{(n-1)}$  is less than the "correlation coefficient" between  $Y_{(2)}$  and  $Y_{(n)}$ .

Theorem 2 tells us the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  of the exponential random variables is always less than the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  of the uniform random variables. From our computation of a few continuous random variables, it looks like this property seems to hold. So we make the following conjecture.

Conjecture II:

Theorem 2 holds for any continuous random variables, i.e., the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  of any continuous random variables is always less than the "correlation coefficient" between  $X_{(k)}$  and  $X_{(k+t)}$  of the uniform random variables, here  $k, t$  are positive integers and  $k+t \leq n, n \geq 2$ .

## References

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